DEFINITION C.10 *A point* $\mathbf{x}^*$ *satisfying* $\mathbf{g}(\mathbf{x}^*) \leq \mathbf{d}$ *is said to be a* **regular point** *of the constraints* $\mathbf{g}(\mathbf{x}) \leq \mathbf{d}$ *if the vectors in the set* $\{\nabla \mathbf{g}_j(\mathbf{x}^*) : g_j(\mathbf{x}^*) = d_j\}$ *are linearly independent,*

This leads to the following first-order necessary conditions (called the *Kuhn-Tucker conditions*):

PROPOSITION C.10 *Suppose* $f \in C^1$ *and* $\mathbf{x}^*$ *is a local maximum of the function* f *over the constraint set* $X = \{\mathbf{x} : \mathbf{g}(\mathbf{x}) \leq \mathbf{d}\}$. *Then if* $\mathbf{x}^*$ *is a regular point, there exist a vector* $\boldsymbol{\pi} \in \Re^m$ *with* $\boldsymbol{\pi} \geq 0$ *such that*

$$\begin{aligned} \nabla \mathbf{f}(\mathbf{x}^*) - \boldsymbol{\pi}^\top \nabla \mathbf{g}(\mathbf{x}^*) &= 0 \\ \boldsymbol{\pi}^\top (\mathbf{d} - \mathbf{g}(\mathbf{x}^*)) &= 0. \end{aligned}$$

In the convex case, both Propositions C.9 and C.10 provide sufficient conditions for optimality. That is, if f is concave, the set X defined by the equality or inequality constraints is convex, and we find a feasible solution $\mathbf{x}^*$ and an associated multiplier $\boldsymbol{\pi}$ satisfy the conditions of Propositions C.9 (or Proposition C.10 in the inequality case), then $\mathbf{x}^*$ is a global maximum.

Sensitivity Analysis

The Lagrange multipliers have an interpretation as giving the rate of change of the objective function as a function of the right-hand side vectors. Indeed, let

$$\begin{aligned} v(\mathbf{b}) \equiv \quad & \max \quad f(\mathbf{x}) \\ & \text{s.t.} \quad \mathbf{h}(\mathbf{x}) = \mathbf{b}. \end{aligned}$$

Then under some relatively mild regularity conditions (see Bertsekas [59]), one can show

$$\nabla_b \mathbf{v}(\mathbf{b}) = \boldsymbol{\pi},$$

where $\boldsymbol{\pi}$ is the Lagrange multiplier associated with an equality-constrained optimal solution $\mathbf{x}^*$. Similarly,if

$$\begin{aligned} v(\mathbf{d}) \equiv \quad & \max \quad f(\mathbf{x}) \\ & \text{s.t.} \quad \mathbf{g}(\mathbf{x}) \leq \mathbf{d}, \end{aligned}$$

then

$$\nabla_d \mathbf{v}(\mathbf{d}) = \boldsymbol{\pi},$$

where $\boldsymbol{\pi} \geq \mathbf{0}$ is the Lagrange multiplier associated with the corresponding optimal solution $\mathbf{x}^*$. The multipliers therefore measure the effect that small changes in the right-hand-sides have on the optimal objective function value.

Parametric Monotonicity

Paramteric monotonicity addresses the question of how optimal solutions vary as a function of the parameters of an optimization problem. These parametric monotonicity results are used, for example, in the analysis of the base-stock, list price policies of